Asymptotic dynamics of the exceptional Bianchi cosmologies

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Abstract. In this paper we give, for the first time, a qualitative description of the asymptotic dynamics of a class of non-tilted spatially homogeneous (SH) cosmologies, the so-called exceptional Bianchi cosmologies, which are of Bianchi type $VI_{-1/9}$. This class is of interest for two reasons. Firstly, it is generic within the class of non-tilted SH cosmologies, being of the same generality as the models of Bianchi types VIII and IX. Secondly, it is the SH limit of a generic class of spatially inhomogeneous G_2 cosmologies.

Using the orthonormal frame formalism and Hubble-normalized variables, we show that the exceptional Bianchi cosmologies differ from the non-exceptional Bianchi cosmologies of type VI_h in two significant ways. Firstly, the models exhibit an oscillatory approach to the initial singularity and hence are not asymptotically self-similar. Secondly, at late times, although the models are asymptotically self-similar, the future attractor for the vacuum-dominated models is the so-called Robinson-Trautman SH model instead of the vacuum SH plane wave models.

1. Introduction

In this paper we give a qualitative description of the asymptotic dynamics of a class of spatially homogeneous (SH) cosmological models which to date have received little attention in the literature. The matter content is a perfect fluid with equation of state $p = (\gamma - 1)\mu$, where γ is a constant. The models are *non-tilted*, i.e. the fluid 4-velocity is orthogonal to the orbits of the 3-parameter group G_3 of isometries, that is of Bianchi type VI_h, with the value of the group parameter $h = -\frac{1}{9}$. This exceptional value permits the shear tensor to have an additional independent component (see Ellis and MacCallum 1969, p 123, case Bbii) with the result that this class of models depends on five arbitrary parameters instead of four, making it of maximum generality, i.e. of the same generality as the non-tilted Bianchi VIII and Bianchi IX models. We shall refer to this class of SH cosmologies as the *exceptional Bianchi cosmologies*.

The longer term goal in studying the dynamics of SH cosmologies is to gain insight into the dynamics of spatially inhomogeneous cosmologies. Indeed our interest in the exceptional Bianchi cosmologies stems from the fact that they are closely related to the simplest class of spatially inhomogeneous cosmologies, the so-called G_2 cosmologies, which admit a 2-parameter Abelian isometry group G_2. The relation is due to the fact that the present class† admits an Abelian G_2 of isometries as a subgroup of the full

† Models of all Bianchi types except for types VIII and IX have this property.

isometry group. Most research on G_2 cosmologies has dealt with two special cases, either diagonal G_2 cosmologies, in which case the line element is diagonal, or more generally, orthogonally transitive G_2 cosmologies, in which case the line element has a 2×2 block form (see van Elst *et al* 2002, where other references can be found). The exceptional Bianchi cosmologies are unique among the non-tilted SH cosmologies in that *the G_2 subgroup does not act orthogonally transitively*, and, as a result, this class of models is the SH limit† of the most general G_2 cosmologies. One thus expects that the asymptotic dynamics of the present class of SH cosmologies will provide insight into the dynamics of general G_2 cosmologies.

In this paper we show that the presence of the additional shear degree of freedom has a significant effect on the asymptotic dynamics of the exceptional Bianchi cosmologies, both in the singular regime and in the late-time regime. Firstly, this shear degree of freedom destabilizes the vacuum Kasner solutions, leading to an oscillatory singularity. Secondly, it destabilizes the SH vacuum plane wave solutions, with the result that they no longer play a dominant role as regards the late-time dynamics.

The plan of this paper is as follows. In section 2 we present the evolution equations for the exceptional Bianchi cosmologies, using the orthonormal frame formalism, and discuss some properties of the Hubble-normalized state space. Details of the derivation are provided in appendix A. In section 3 we give the equilibrium points of the evolution equations, and describe their local stability, indicating the various bifurcations that are determined by the equation of state parameter γ. Sections 4 and 5 contain the main results of the paper, a description of the asymptotic dynamics of the models at late times and near the initial singularity, in terms of the future attractor and the past attractor of the evolution equations. Finally, in section 6 we conclude by pointing out some analogies between the exceptional Bianchi cosmologies and other classes of SH cosmologies with an oscillatory singularity. We also comment on the implications of our results for G_2 cosmologies.

It is assumed that the reader is familiar both with the orthonormal frame formalism of Ellis and MacCallum (see MacCallum 1973), and with the use of dynamical systems techniques in cosmology, for which we refer the reader to the book of Wainwright and Ellis (1997)‡. We use geometrized units with $8\pi G = c = 1$ and the sign conventions of MacCallum (1973).

2. Evolution equations

In this section we give the evolution equations for the exceptional Bianchi cosmologies. The equations are derived in appendix A using the orthonormal frame formalism and Hubble-normalized variables. The models are described by a dimensionless state vector

$$\boldsymbol{x} = (\Sigma_+, \Sigma_-, \Sigma_2, \Sigma_\times, N_-, A), \tag{2.1}$$

subject to a constraint given by

$$g(\boldsymbol{x}) = (\Sigma_+ + \sqrt{3}\,\Sigma_-)A - \Sigma_\times N_- = 0. \tag{2.2}$$

The variables Σ_+, Σ_-, Σ_2 and $\Sigma_\times$ describe the shear of the fluid congruence, while N_- and A describe the spatial curvature. These variables are dimensionless, having

† There are also classes of *tilted* SH models that are SH limits of the most general G_2 cosmologies (see Wainwright and Ellis 1997, table 12.4, p 268).
‡ Henceforth, we will refer to this work as WE.

been normalized with the Hubble scalar† H, which is related to the overall length scale ℓ by

$$H = \frac{\dot{\ell}}{\ell}. \tag{2.3}$$

The overdot denotes differentiation with respect to clock time along the fluid congruence. The state variables depend on a dimensionless time variable τ that is related to the length scale ℓ by

$$\ell = \ell_0 e^\tau, \tag{2.4}$$

where ℓ_0 is a constant. The dimensionless time τ is related to the clock time t by

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{H}, \tag{2.5}$$

as follows from equations (2.3) and (2.4). In formulating the evolution equations we require the deceleration parameter q, defined by

$$q = -\frac{\ell\ddot{\ell}}{\dot{\ell}^2}, \tag{2.6}$$

and the density parameter Ω, defined by

$$\Omega = \frac{\mu}{3H^2}. \tag{2.7}$$

The evolution equations for the components of $\boldsymbol{x}$ in (2.1) take the following form

$$\begin{aligned}
\Sigma'_+ &= (q-2)\Sigma_+ + 3\Sigma_2^2 - 2N_-^2 - 6A^2, \\
\Sigma'_- &= (q-2)\Sigma_- - \sqrt{3}\,\Sigma_2^2 + 2\sqrt{3}\,\Sigma_\times^2 - 2\sqrt{3}\,N_-^2 + 2\sqrt{3}\,A^2, \\
\Sigma'_2 &= (q - 3\Sigma_+ + \sqrt{3}\,\Sigma_- - 2)\Sigma_2, \\
\Sigma'_\times &= (q - 2\sqrt{3}\,\Sigma_- - 2)\Sigma_\times - 8N_- A, \\
N'_- &= (q + 2\Sigma_+ + 2\sqrt{3}\,\Sigma_-)N_- + 6\Sigma_\times A \\
A' &= (q + 2\Sigma_+)A,
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
q &= 2\Sigma^2 + \tfrac{1}{2}(3\gamma - 2)\Omega, \tag{2.9} \\
\Omega &= 1 - \Sigma^2 - N_-^2 - 4A^2, \tag{2.10} \\
\Sigma^2 &= \Sigma_+^2 + \Sigma_-^2 + \Sigma_2^2 + \Sigma_\times^2, \tag{2.11}
\end{aligned}$$

and $'$ denotes differentiation with respect to τ. There are two auxiliary equations, the evolution equation for Ω, which reads

$$\Omega' = [2q - (3\gamma - 2)]\Omega, \tag{2.12}$$

and the evolution equation for the constraint function $g(\boldsymbol{x})$, namely,

$$g' = 2(q + \Sigma_+ - 1)g, \tag{2.13}$$

which guarantees that $g(\boldsymbol{x}) = 0$ defines an invariant set. Both (2.12) and (2.13) follow from (2.8).

† On account of (2.3), H is related to the rate of volume expansion Θ of the fluid congruence according to $H = \frac{1}{3}\Theta$.

The physical state space is the subset of $\mathbb{R}^6$ defined by the constraint (2.2) and the requirement

$$\Omega \geqslant 0. \tag{2.14}$$

The restriction (2.14) implies that the state space is bounded. In addition, the evolution equations are invariant under the symmetries

$$(\Sigma_+, \Sigma_-, \Sigma_2, \Sigma_\times, N_-, A) \longrightarrow (\Sigma_+, \Sigma_-, \pm\Sigma_2, \pm\Sigma_\times, \pm N_-, \pm A), \tag{2.15}$$

provided that the product $\Sigma_\times N_- A$ does not change sign. Since the evolution equations for Σ_2 and A do not allow either of these variables to change sign along an orbit, we can assume without loss of generality that

$$\Sigma_2 \geqslant 0, \qquad A \geqslant 0. \tag{2.16}$$

Thus the physical state space $\mathcal{D}$ is the subset $\mathcal{D} \subset \mathbb{R}^6$ defined by (2.2), (2.14) and (2.16).

The state space $\mathcal{D}$ contains a number of invariant subsets which play an important role as regards the dynamics in the asymptotic regimes. Firstly there is the *vacuum boundary*, given by $\Omega = 0$. Secondly there are various invariant sets with $A = 0$ that describe vacuum Bianchi I (Kasner) and vacuum Bianchi II (Taub) models, which we shall introduce in section 5. Thirdly, it follows from the evolution equations (2.8) that the conditions†

$$\Sigma_\times = 0, \qquad N_- = 0, \qquad \Sigma_+ + \sqrt{3}\,\Sigma_- = 0, \tag{2.17}$$

define a 3-dimensional invariant set, which we shall denote by S. This set describes the subclass of the exceptional Bianchi cosmologies for which the G_2 admits one hypersurface-orthogonal Killing vector field. The asymptotic dynamics of this subclass have been analyzed in detail by Hewitt (1991). Finally, we note that the invariant set $\Sigma_2 = 0$ describes the non-exceptional Bianchi VI$_{-1/9}$ models. In other words, *the variable Σ_2 describes the additional shear degree of freedom referred to in the introduction.*

3. Local stability of the equilibrium points

In this section we discuss the local stability of the equilibrium points of the DE (2.8) subject to the constraint (2.2). The equilibrium points can be found by solving the system of algebraic equations

$$\boldsymbol{f}(\boldsymbol{a}) = \boldsymbol{0}, \qquad g(\boldsymbol{a}) = 0.$$

The local stability is determined by linearizing the DE (2.8) at $\boldsymbol{x} = \boldsymbol{a}$, which gives

$$\boldsymbol{x}' = \mathrm{D}\boldsymbol{f}(\boldsymbol{a})\boldsymbol{x},$$

and finding the eigenvalues of the derivative matrix $\mathrm{D}\boldsymbol{f}(\boldsymbol{a})$. The existence of the constraint (2.2) complicates the analysis, which requires that we consider only the physical eigenvectors of $\mathrm{D}\boldsymbol{f}(\boldsymbol{a})$, that is, those which are tangent to the constraint surface, or equivalently, those which are orthogonal to the gradient vector $\nabla g(\boldsymbol{a})$. Eigenvalues and eigenvectors that satisfy this condition shall be referred to as *physical*. We note that if all the physical eigenvalues have negative (positive) real parts then the equilibrium point is a local sink (source), that is, it attracts (repels) all orbits in

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† On account of the constraint (2.2) the first two restrictions imply that $\Sigma_+ + \sqrt{3}\,\Sigma_- = 0$, since $A \neq 0$ for the exceptional Bianchi cosmologies.

a neighbourhood. In addition to isolated equilibrium points, we will also encounter arcs of equilibrium points, for which one physical eigenvalue is necessarily zero. In this case, the criterion for a local sink (source) is that all eigenvalues other than the zero one have negative (positive) real parts.

We now list the equilibrium points of the DE (2.8) subject to the constraint (2.2), and give the values of the density parameter Ω and the deceleration parameter q. We refer to WE (see pp 189–93) for the corresponding exact solutions.

Non-vacuum equilibrium points ($\Omega > 0$)

3.1. Flat Friedmann-Lemaître equilibrium point, FL

$$\Sigma_+ = \Sigma_- = \Sigma_2 = \Sigma_\times = N_- = A = 0,$$
$$\Omega = 1, \qquad q = \tfrac{1}{2}(3\gamma - 2), \qquad 0 < \gamma < 2.$$

3.2. Collins-Stewart Bianchi II equilibrium points, CS$^\pm$

$$\Sigma_+ = -\tfrac{1}{16}(3\gamma - 2), \qquad \Sigma_- = \sqrt{3}\,\Sigma_+, \qquad N_- = \pm\tfrac{1}{8}\sqrt{3(3\gamma - 2)(2 - \gamma)},$$
$$\Sigma_2 = \Sigma_\times = A = 0,$$
$$\Omega = \tfrac{3}{16}(6 - \gamma), \qquad q = \tfrac{1}{2}(3\gamma - 2), \qquad \tfrac{2}{3} < \gamma < 2.$$

3.3. Collins Bianchi VI$_{-1/9}$ equilibrium point, C

$$\Sigma_+ = -\tfrac{1}{4}(3\gamma - 2), \qquad \Sigma_- = -\tfrac{1}{\sqrt{3}}\Sigma_+, \qquad A = \tfrac{1}{4}\sqrt{(3\gamma - 2)(2 - \gamma)},$$
$$\Sigma_2 = \Sigma_\times = N_- = 0,$$
$$\Omega = \tfrac{1}{3}(5 - 3\gamma), \qquad q = \tfrac{1}{2}(3\gamma - 2), \qquad \tfrac{2}{3} < \gamma < \tfrac{5}{3}.$$

3.4. Wainwright Bianchi VI$_{-1/9}$ arc of equilibrium points, W

$$\Sigma_+ = -\tfrac{1}{3}, \qquad \Sigma_- = \tfrac{1}{3\sqrt{3}}, \qquad \Sigma_2 = \tfrac{\sqrt{5}\,\alpha}{3\sqrt{3}},$$
$$\Sigma_\times = N_- = 0, \qquad A = \sqrt{\tfrac{1}{54}(4 + 5\alpha^2)}, \qquad 0 < \alpha < 1,$$
$$\Omega = \tfrac{5}{9}(1 - \alpha^2), \qquad q = \tfrac{2}{3}, \qquad \gamma = \tfrac{10}{9}.$$

Vacuum equilibrium points ($\Omega = 0$)

3.5. Robinson-Trautman Bianchi VI$_{-1/9}$ equilibrium point, RT

$$\Sigma_+ = -\tfrac{1}{3}, \qquad \Sigma_- = \tfrac{1}{3\sqrt{3}}, \qquad \Sigma_2 = \tfrac{\sqrt{5}}{3\sqrt{3}},$$
$$\Sigma_\times = N_- = 0, \qquad A = \tfrac{1}{\sqrt{6}},$$
$$q = \tfrac{2}{3}.$$

Table 1. Local sinks in the exceptional Bianchi state space.

Range of γ	Local sink	Ω	Σ^2	$\mathcal{W}$
$0 < \gamma \leqslant \frac{2}{3}$	FL	1	0	0
$\frac{2}{3} < \gamma < \frac{10}{9}$	C	$\frac{1}{3}(5 - 3\gamma)$	$\frac{1}{12}(3\gamma - 2)^2$	$\frac{1}{6}(3\gamma - 2)\sqrt{3\gamma + 1}$
$\gamma = \frac{10}{9}$	W	$\frac{5}{9}(1 - \alpha^2)$	$\frac{1}{27}(4 + 5\alpha^2)$	$\frac{1}{9\sqrt{3}}\sqrt{(4 + 5\alpha^2)(13 + 35\alpha^2)}$
$\frac{10}{9} < \gamma < 2$	RT	0	$\frac{1}{3}$	$\frac{4}{3}$

3.6. Plane wave arcs of equilibrium points, $\mathsf{PW}^{\pm}$

$$\Sigma_+ = -\tfrac{1}{4}(4 - \alpha), \qquad \Sigma_- = \tfrac{1}{4}\sqrt{3}\,\alpha, \qquad \Sigma_2 = 0,$$
$$\Sigma_\times = \pm\tfrac{1}{2}\sqrt{\alpha(1 - \alpha)}, \qquad N_- = -\Sigma_\times, \qquad A = \tfrac{1}{4}\alpha, \qquad 0 < \alpha \leqslant 1,$$
$$q = \tfrac{1}{2}(4 - \alpha).$$

3.7. Kasner circle of equilibrium points, $\mathcal{K}$

$$\Sigma_2 = \Sigma_\times = N_- = A = 0, \qquad \Sigma_+^2 + \Sigma_-^2 = 1,$$
$$q = 2.$$

We now consider whether any of the equilibrium points are local sinks or sources. It turns out that for each value of the equation of state parameter γ in the interval $0 < \gamma < 2$, there is a unique equilibrium point/set that is a local sink, as indicated in table 1. We can describe the transitions, i.e. bifurcations, that occur between these local sinks as follows:

$$\mathsf{FL} \xrightarrow{\gamma = \frac{2}{3}} \mathsf{C} \xrightarrow{\gamma = \frac{10}{9}} \mathsf{W} \xrightarrow{\gamma = \frac{10}{9}} \mathsf{RT} \tag{3.1}$$

The mechanisms for these bifurcations, without giving full details, can be described as follows. The linearization of the evolution equations for N_- and A at the flat FL point is

$$N_-' = \tfrac{1}{2}(3\gamma - 2)N_-, \qquad A' = \tfrac{1}{2}(3\gamma - 2)A,$$

establishing that the spatial curvature variables N_- and A destabilize FL at $\gamma = \frac{2}{3}$. Moreover when $\gamma = \frac{2}{3}$, both the Collins-Stewart CS and Collins C equilibrium points bifurcate from the FL equilibrium point. The linearization of the evolution equation for Σ_2 at the Collins equilibrium point C is

$$\Sigma_2' = \tfrac{1}{2}(9\gamma - 10)\Sigma_2,$$

showing that Σ_2 destabilizes C at $\gamma = \frac{10}{9}$. Indeed, at $\gamma = \frac{10}{9}$ there is a line bifurcation which is characterized by the exchange of stability between the Collins equilibrium point C and the Robinson-Trautman vacuum equilibrium point RT by means of the Wainwright arc of equilibria W which connects both points. It is also of interest to note that for $\frac{10}{9} < \gamma < \frac{5}{3}$ C is a saddle, and it merges with the plane wave arcs (at $\alpha = 1$) when $\gamma = \frac{5}{3}$.

The local sinks in the exceptional Bianchi state space are listed in table 1, together with the values of the density parameter Ω, the shear parameter Σ, which describes the anisotropy in the fluid congruence (see WE, p 114) and the Weyl curvature parameter $\mathcal{W}$, which can be regarded as a measure of the intrinsic anisotropy of the gravitational

field (see Wainwright *et al* 1999, p 2580). For the present class of models, Ω and Σ^2 are given by (2.10) and (2.11). The formula for $\mathcal{W}$ is provided in appendix B.

It is of interest to note that the plane wave arcs of equilibrium points, $\mathsf{PW}^{\pm}$, which describe the SH plane wave vacuum solutions, do not appear in the table. This result represents an important difference between the exceptional and non-exceptional Bianchi VI_h models. For the latter class, the plane wave equilibrium points are a local sink for vacuum models and for perfect fluid models, subject to a restriction on the equation of state parameter γ. On the other hand, the plane wave arcs of equilibria are saddles in the exceptional Bianchi state space. To see this, we linearize the evolution equation for Σ_2 at the plane wave arcs obtaining

$$\Sigma_2' = \tfrac{1}{2}(6 - \alpha)\Sigma_2.$$

Thus, *the extra shear degree of freedom, Σ_2, that is present in the exceptional Bianchi cosmologies, destabilizes the plane wave solutions.* Their stability is, in effect, inherited by the Robinson-Trautman solution.

As regards local sources, it follows upon analysis of the eigenvalues associated with the equilibrium points that *there are no local sources in the exceptional Bianchi state space.* In particular, the Kasner circle $\mathcal{K}$ is a saddle, and, consequently, the initial state of a typical model cannot be a single Kasner equilibrium point.

4. The late-time asymptotic regime

We have seen that for each value of γ in the range $0 < \gamma < 2$, excluding $\gamma = \frac{10}{9}$, there is a unique equilibrium point that is a local sink of the evolution equations, while if $\gamma = \frac{10}{9}$, the local sink is an arc of equilibrium points. We note that all of the local sinks in table 1 lie in the three-dimensional invariant set S defined by (2.17). It has been proved that each of these local sinks is the future attractor in the invariant set S, for the relevant range of γ (see Hewitt 1991)†.

We have performed extensive numerical experiments which suggest that the solution determined by an arbitrary initial condition satisfies

$$\lim_{\tau \to +\infty} \Sigma_\times = 0, \qquad \lim_{\tau \to +\infty} N_- = 0, \qquad \lim_{\tau \to +\infty} \Sigma_+ + \sqrt{3}\,\Sigma_- = 0. \qquad (4.1)$$

In other words, the experiments provide strong evidence that all orbits are attracted to the invariant set S, i.e. that the future attractor is contained in this subset. We thus make the following conjecture.

Conjecture 4.1. *Each local sink given in table 1 for γ in the range $0 < \gamma < 2$ is the future attractor in the exceptional Bianchi state space.*

This conjecture is in fact known to be true for γ in the range $0 < \gamma < \frac{2}{3}$ (see WE, theorem 8.2, p 174). Giving a proof for γ in the range $\frac{2}{3} \leqslant \gamma < 2$ necessitates establishing the limits (4.1).

5. The singular asymptotic regime

In this section we show that the exceptional Bianchi cosmologies exhibit an oscillatory approach to the initial singularity, as do the Bianchi VIII and IX models. There are two significant differences, however, which we now describe.

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† The analysis in this paper is complete, except for the case $\gamma = \frac{10}{9}$. We have recently been able to complete this analysis by finding a Dulac function for the family of two-dimensional invariant subsets that foliate the state space. We thank Neville Dubash for assistance with this matter.
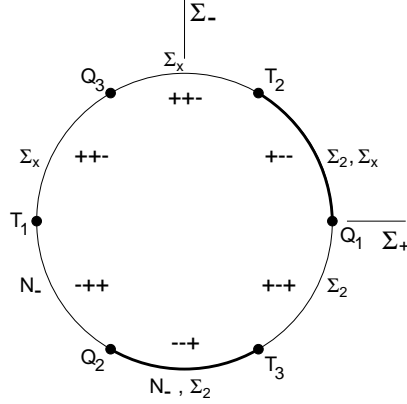
Figure 1. The arrays $+ + -$, etc. give the signs of the eigenvalues λ_{N_-}, λ_{Σ_2} and $\lambda_{\Sigma_\times}$ in that order. The variables(s) listed next to each of the six arcs indicates which of the variable(s) N_-, Σ_2, $\Sigma_\times$ is growing into the past. Kasner points on the bold arcs T_2Q_1 and T_3Q_2 have a two-dimensional Kasner transition set into the past.

In the dynamical systems approach, the mechanism for creating an oscillatory singularity is that the Kasner equilibrium points, which comprise the Kasner circle, are saddles, and the unstable manifold (into the past) of any Kasner point is asymptotic to another Kasner point. In other words, the unstable manifold consists of orbits that join two Kasner points. We shall refer to these unstable manifolds as *Kasner transition sets*, because they provide a mechanism for a cosmological model to make a transition from one (approximate) Kasner state to another, as it evolves into the past. For non-tilted Bianchi VIII and IX models, each Kasner transition set is *one-dimensional* and thus there is a unique heteroclinic orbit† joining two Kasner points. In the present case, for two of the six arcs on the Kasner circle, the Kasner transition set is *two-dimensional*, leading to a one-parameter family of heteroclinic orbits. Nevertheless, in both cases, the stability properties of the Kasner equilibrium points lead to the existence of infinite heteroclinic sequences, i.e. infinite sequences of equilibrium points on the Kasner circle joined by special heteroclinic orbits, oriented into the past (see WE, section 6.4.2, for Bianchi VIII and IX models). These heteroclinic sequences govern the dynamics in the singular regime ($\tau \to -\infty$) in the sense that a typical orbit shadows (i.e. is approximated by) a heteroclinic sequence as $\tau \to -\infty$. In physical terms, the corresponding cosmological model is approximated by a sequence of Kasner vacuum models as the singularity is approached into the past, the so-called *Mixmaster oscillatory singularity*.

The second difference is that in the case of Bianchi VIII and IX models, the instability of the Kasner points is generated by spatial curvature only, while in the case of exceptional Bianchi cosmologies, the instability is due to a combination of spatial curvature and off-diagonal shear, in particular, the variables N_-, Σ_2 and $\Sigma_\times$.

Figure 1 shows the signs of the eigenvalues of the Kasner equilibrium points associated with the variables N_-, Σ_2 and $\Sigma_\times$, showing that points on the arcs T_2Q_1 and T_3Q_2 have a two-dimensional Kasner transition set into the past (negative eigenvalues indicate instability into the past). The figure also shows which of the three variables are increasing into the past in a neighbourhood of the Kasner circle.

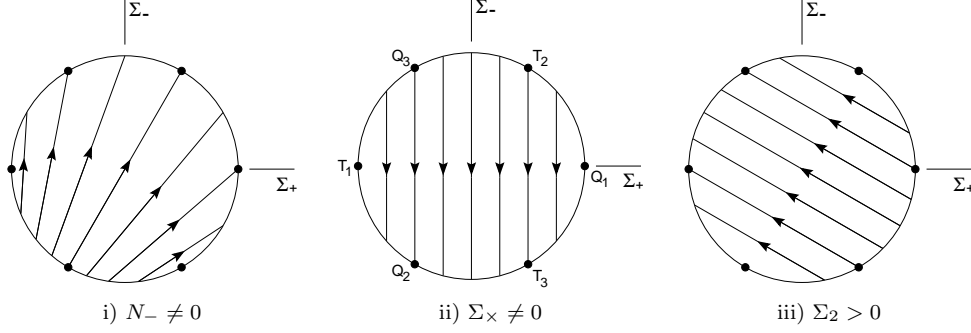† A heteroclinic orbit is one that joins two equilibrium points.

i) $N_- \neq 0$ ii) $\Sigma_\times \neq 0$ iii) $\Sigma_2 > 0$

Figure 2. Vacuum orbits joining points on the Kasner circle $\mathcal{K}$. The arrows show evolution into the past.

Figure 2 shows the projections in the $\Sigma_+\Sigma_-$-plane, of three families of heteroclinic orbits, that join two Kasner points. For a Kasner point with a one-dimensional transition set, the orbit shown in figure 2 emanating from that point is the transition set. These families are described as follows.

$\mathcal{S}_{N_-}$: *Vacuum Bianchi II models (Taub models)*

These orbits describe the familiar vacuum Bianchi II models. They are given by

$$\Sigma_2 = \Sigma_\times = A = 0, \qquad \Sigma_+^2 + \Sigma_-^2 + N_-^2 = 1, \qquad N_- \neq 0, \tag{5.1}$$

with $\Sigma_- + \sqrt{3} = m(\Sigma_+ + 1)$, where $m > \frac{1}{\sqrt{3}}$ is a constant.

$\mathcal{S}_{\Sigma_\times}$: *Kasner models relative to a non-Fermi-propagated frame, with $\Sigma_2 = 0$*

These orbits are given by

$$\Sigma_2 = N_- = A = 0, \qquad \Sigma_+^2 + \Sigma_-^2 + \Sigma_\times^2 = 1, \qquad \Sigma_\times \neq 0, \tag{5.2}$$

with $\Sigma_+ = m$, where $-1 < m < 1$ is a constant.

$\mathcal{S}_{\Sigma_2}$: *Kasner models relative to a non-Fermi-propagated frame, with $\Sigma_\times = 0$*

These orbits are given by

$$\Sigma_\times = N_- = A = 0, \qquad \Sigma_+^2 + \Sigma_-^2 + \Sigma_2^2 = 1, \qquad \Sigma_2 > 0, \tag{5.3}$$

with $\Sigma_+ + \sqrt{3}\,\Sigma_- = m$, where $-2 < m < 2$ is a constant.

Figure 3 gives a representation of the two-dimensional transition set of a Kasner point on the arc T_3Q_2. This invariant set consists of a one-parameter family of orbits with $\Omega = 0$, $A = 0$, $\Sigma_\times = 0$, $N_- \neq 0$ and $\Sigma_\times \neq 0$, which join A to B (the projection of two typical orbits is shown), and two special orbits, one with $N_- \neq 0$ and $\Sigma_2 = \Sigma_\times = 0$ (a member of the family $\mathcal{S}_{N_-}$), and the other with $\Sigma_2 \neq 0$ and $N_- = \Sigma_\times = 0$ (a member of the family $\mathcal{S}_{\Sigma_2}$). These orbits, which describe the vacuum Bianchi II
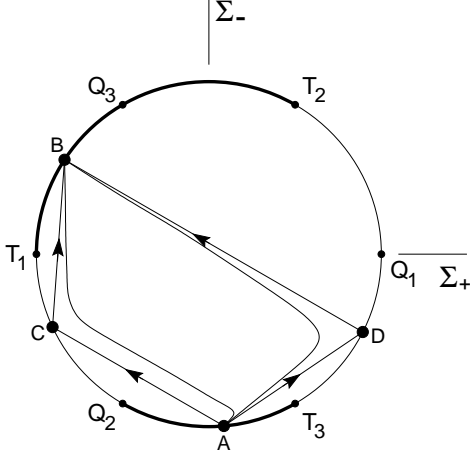
Figure 3. The two-dimensional Kasner transition set of a Kasner point A on the arc T_3Q_2, projected onto the $\Sigma_+\Sigma_-$-plane. The arrows show evolution into the past.
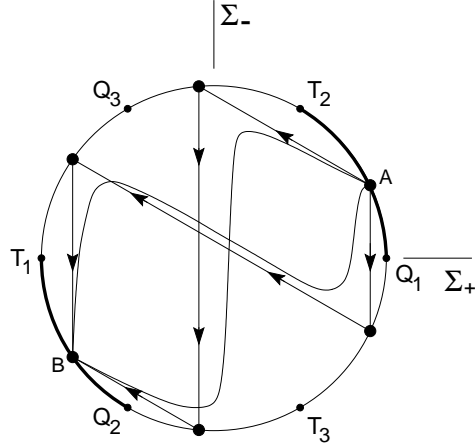
Figure 4. The two-dimensional Kasner transition set of a Kasner point A on the arc T_2Q_1, projected onto the $\Sigma_+\Sigma_-$-plane. The arrows show evolution into the past.

models relative to a non-Fermi-propagated frame, can be described explicitly by the equation†

$$\frac{(1 + \Sigma_+ + \sqrt{3}\,\Sigma_-)^2 + 3N_-^2}{(4 + \Sigma_+ + \sqrt{3}\,\Sigma_-)^2} = \alpha, \tag{5.4}$$

where α is a constant that depends on the point A.

Figure 4 gives a representation of the two-dimensional transition set of a Kasner point on the arc T_2Q_1. This invariant set consists of a one-parameter family of orbits with $\Omega = 0$, $A = 0$, $N_- = 0$, $\Sigma_2 \neq 0$ and $\Sigma_\times \neq 0$, which join A to B (the projection of two typical orbits is shown), and two special orbits, one with $\Sigma_2 \neq 0$ and $\Sigma_\times = N_- = 0$ (a member of the family $\mathcal{S}_{\Sigma_2}$), and the other with $\Sigma_\times \neq 0$ and $N_- = \Sigma_2 = 0$ (a member of the family $\mathcal{S}_{\Sigma_\times}$). All orbits in this manifold, which intersect the Kasner circle in six points, as shown in figure 4, represent one and the same physical Kasner model. The six Kasner points referred to have the same Kasner exponents, but differ in the labelling, and the orbits in the Kasner transition set thus describe a rotation of the spatial axes relative to which this Kasner model is described. These orbits can be described explicitly by the equation‡

$$2 - 6\Sigma_+ + 8\Sigma_+^3 + 3\sqrt{3}(\sqrt{3}\,\Sigma_+ + \Sigma_-)\Sigma_2^2 = \alpha, \tag{5.5}$$

where α is a constant that depends on the point A, thereby distinguishing different Kasner models.

Numerical experiments suggest that as $\tau \to -\infty$, an orbit that is close to a point A on the arc T_3Q_2 will shadow either the orbit $AD \subset \mathcal{S}_{N_-}$ or the orbit $AC \subset \mathcal{S}_{\Sigma_2}$,

† This expression is a first integral of the evolution equations (2.8) when restricted to the invariant set $\Omega = 0$, $A = 0$, $\Sigma_\times = 0$. An equivalent expression has arisen in the analysis of G_2 cosmologies (see Weaver 2002). We thank Marsha Weaver for pointing out this first integral in the G_2 context, and Woei Chet Lim for calculating the expression in (5.4).

‡ This expression is a first integral of the evolution equations (2.8) when restricted to the invariant set $\Omega = 0$, $A = 0$, $N_- = 0$. It is related to a Hubble-normalized scalar formed from the Weyl tensor, which is constant for Kasner models. We thank Woei Chet Lim for verifying this result.

rather than shadowing a typical orbit in the two-dimensional Kasner transition set. A similar remark applies to points on the arc T_2Q_1. The result of this behaviour is that as $\tau \to -\infty$ an orbit will shadow a heteroclinic sequence that consists of orbits belonging to the invariant sets $\mathcal{S}_{N_-}$, $\mathcal{S}_{\Sigma_2}$ and $\mathcal{S}_{\Sigma_\times}$. This behaviour motivates the following conjecture concerning the past attractor $\mathcal{A}^-$.

Conjecture 5.1. *The past attractor is the two-dimensional invariant set consisting of all orbits in the invariant sets $\mathcal{S}_{N_-}$, $\mathcal{S}_{\Sigma_\times}$ and $\mathcal{S}_{\Sigma_2}$ (see figure 2) and the Kasner equilibrium points, i.e.*

$$\mathcal{A}^- = \mathcal{S}_{N_-} \cup \mathcal{S}_{\Sigma_\times} \cup \mathcal{S}_{\Sigma_2} \cup \mathcal{K}. \tag{5.6}$$

This conjecture can be formulated in terms of limits of the state variables as follows. Referring to (5.1)–(5.3), (2.10) and (2.11) we see that the set $\mathcal{A}^-$ is defined by

$$\Omega = 0, \qquad A = 0$$

and

$$N_- \Sigma_2 = 0, \qquad \Sigma_2 \Sigma_\times = 0, \qquad \Sigma_\times N_- = 0,$$

the final equality being a consequence of $A = 0$ and the constraint (2.2). Thus our conjecture concerning the past attractor can be formulated as

$$\lim_{\tau \to -\infty} \Omega = 0, \qquad \lim_{\tau \to -\infty} \Delta = 0,$$

where

$$\Delta = (N_- \Sigma_2)^2 + (\Sigma_2 \Sigma_\times)^2 + (\Sigma_\times N_-)^2.$$

These limits imply that $\lim_{\tau \to -\infty} A = 0$, on account of (2.2). Note that for a generic orbit, $\lim_{\tau \to -\infty}(N_-, \Sigma_2, \Sigma_\times)$ does not exist.

6. Discussion

There are two generic classes of *non-tilted* ever-expanding SH cosmologies, namely

  i) the Bianchi type VIII cosmologies, and
 ii) the exceptional Bianchi cosmologies.

The evolution of the first class in the asymptotic regimes has been described in the literature (see for example, WE, pp 143–7 for the singular regime, and the accompanying paper, Horwood *et al* 2002, for the late-time regime). In this paper we have used Hubble-normalized variables to describe the asymptotic dynamics of the second class. We have shown that with this choice of variables, the Einstein field equations reduce to an autonomous DE on a 5-dimensional compact subset of $\mathbb{R}^6$. Since the state space is compact, there is a past attractor and a future attractor, which we have been able to identify using heuristic arguments and numerical experiments, thereby describing the dynamics in the singular and late-time asymptotic regimes.

The mathematical description of the exceptional Bianchi cosmologies differs from that of the non-exceptional models in the following ways:

  i) the expansion-normalized state space is of dimension five rather than four, corresponding to the presence of an additional shear degree of freedom,
 ii) the Abelian G_2 subgroup of the isometry group does not act orthogonally transitively.

We have shown that these differences lead to significant changes in the asymptotic dynamics:

i) the singular regime is oscillatory (infinite sequence of Kasner states), rather than asymptotically self-similar,

ii) there is a new asymptotic state described by the vacuum SH Robinson-Trautman solution, which describes the late-time behaviour of vacuum models and of perfect fluid models with γ satisfying $\frac{10}{9} < \gamma < 2$.

These two points also highlight the similarities and the differences between the two generic classes of ever-expanding models: both classes have an oscillatory singularity, but differ as regards the late-time asymptotic regime. The Bianchi VIII models are not asymptotically self-similar at late times and exhibit Weyl curvature dominance (the Weyl curvature parameter $\mathcal{W}$ diverges as $\tau \to +\infty$, see Horwood *et al* 2002), whereas the exceptional Bianchi cosmologies are asymptotically self-similar and hence do not exhibit Weyl curvature dominance ($\mathcal{W}$ has a finite limit as $\tau \to +\infty$, given in table 1).

We now discuss some aspects of the oscillatory singular regime that occurs in SH cosmologies. This type of singularity, also known as a Mixmaster singularity, was first discovered in vacuum Bianchi IX cosmologies (see Misner 1969 and Belinskii *et al* 1970) and subsequently in the closely related Bianchi VIII cosmologies (see for example, WE, section 6.4). Within the dynamical systems framework, the oscillatory singularity in these models is created by the three degrees of freedom in the spatial curvature, which destabilize the Kasner equilibrium points in the Hubble-normalized state space. In the present models, there is only one degree of freedom in the spatial curvature, but instead there appear *two off-diagonal shear degrees of freedom*, which again destabilize the Kasner equilibrium points. This destabilization, and the resulting existence of the Kasner transition sets (see section 5), leads to the creation of infinite sequences of heteroclinic orbits joining Kasner equilibrium points, resulting in an oscillatory singularity. This phenomenon can in fact occur in models of all Bianchi types, if one permits sufficient general source terms for the gravitational field. Two classes which have been analyzed in detail and are closely related to the present class, are the Bianchi I magnetic cosmologies (see LeBlanc 1997) which have one magnetic degree of freedom and two off-diagonal shear degrees of freedom, and the tilted perfect fluid Bianchi II cosmologies (see Hewitt *et al* 2001) which have one spatial curvature degree of freedom and two off-diagonal shear degrees of freedom. Indeed, the common feature of all SH models with an oscillatory singularity is the occurrence of the above *Kasner destabilization*, which can be caused by any combination of spatial curvature, off-diagonal shear or magnetic degrees of freedom totaling at least three.

We conclude by discussing the implications of our results for G_2 cosmologies. As mentioned in the introduction, the present class of SH cosmologies are the SH limit of the most general G_2 cosmologies, namely those for which the G_2 does not act orthogonally transitively. One thus expects that the initial singularity in a general G_2 model will be oscillatory, and indeed numerical evidence has been provided that suggests this is the case, at least for vacuum models (see Berger *et al* 2001 and Lim 2002). Secondly, the fact that the SH Robinson-Trautman solution is a local sink in the Hubble-normalized state space for vacuum models and for perfect fluid models with equation of state parameter γ satisfying $\frac{10}{9} < \gamma < 2$ suggests that this solution may play a role in the dynamics of general G_2 cosmologies at late times. It

would thus be of interest to investigate the stability of the SH Robinson-Trautman solution in this context.

Acknowledgments

Appendix A. Derivation of the evolution equations

In this appendix we write the EFEs for the exceptional Bianchi cosmologies as an autonomous system of ordinary differential equations subject to one algebraic constraint. Our starting point is the system of evolution equations for a general spatially homogeneous cosmological model given in Hewitt *et al* (2001) (see equations (A.11)–(A.18)). These equations have been obtained by writing the orthonormal frame equations for a spatially homogeneous model using a group invariant orthonormal frame $\{e_0, e_\alpha\}$, choosing e_0 to be the unit normal to the group orbits. Furthermore, all commutation functions and source terms are normalized with the Hubble scalar H.

We now describe the specializations that lead to the exceptional Bianchi cosmologies. Firstly, the models are non-tilted in the sense that the unit normal e_0 is aligned with the fluid velocity u. Furthermore, the models have a perfect fluid source with a γ-law equation of state given by

$$P = (\gamma - 1)\Omega,$$

where P is the Hubble-normalized pressure and Ω is the dimensionless density parameter. The equation of state parameter γ is restricted to the interval $0 < \gamma < 2$. Secondly, the orthonormal frame vectors e_2 and e_3 are chosen to be tangential to the orbits of the two-dimensional Abelian subgroup G_2 of the three-dimensional symmetry group G_3. The consequences of this frame choice are

$$N_{1\alpha} = 0, \qquad A_\alpha = \delta^1{}_\alpha A_1, \qquad R_3 = -\Sigma_{12}, \qquad R_2 = \Sigma_{13} \qquad \text{(A.1)}$$

(see Wainwright 1979, pp 2019–21). Equation (A.18) in Hewitt *et al* (2001) contains three constraints, one of which will give rise to the constraint (2.2) and the other two which together lead to the following possibilities:

  i) $\Sigma_{12} = \Sigma_{13} = 0$,
    or
  ii) $9A_1^2 - N_{23}^2 + N_{22}N_{33} = 0$ and $N_{22}\Sigma_{12} + (N_{23} - 3A_1)\Sigma_{13} = 0$.

Making the former choice is equivalent to insisting that the Abelian G_2 subgroup acts orthogonally transitively. We make the latter choice, which results in the models being Bianchi VI$_h$ with† $h = -\frac{1}{9}$.

----

† Recall that the group parameter h is related to A_α and $N_{\alpha\beta}$ according to
$$A_\alpha A^\alpha = \tfrac{1}{2}h\left[(N_\alpha{}^\alpha)^2 - N_\alpha{}^\beta N_\beta{}^\alpha\right].$$

The remaining gauge freedom at this point is an initial alignment in the G_2 orbits. We choose to set $\Sigma_{12} = 0$ initially. It follows that this choice is preserved by the evolution equation for Σ_{12} provided that

$$R_1 = \Sigma_{23}. \tag{A.2}$$

The gauge freedom in the angular velocity variables R_α of the spatial frame allows us to make such a choice for R_1. Thus, the constraints in case ii) now yield

$$N_{23} - 3A_1 = 0, \qquad N_{33} = 0. \tag{A.3}$$

At this stage, the independent non-zero Hubble-normalized variables are

$$\Sigma_{22}, \quad \Sigma_{33}, \quad \Sigma_{13}, \quad \Sigma_{23}, \quad N_{22}, \quad A_1.$$

We relabel the shear variables according to

$$\begin{aligned}
\Sigma_+ &= \tfrac{1}{2}(\Sigma_{22} + \Sigma_{33}), & \Sigma_- &= \tfrac{1}{2\sqrt{3}}(\Sigma_{22} - \Sigma_{33}), \\
\Sigma_2 &= \tfrac{1}{\sqrt{3}}\Sigma_{13}, & \Sigma_\times &= \tfrac{1}{\sqrt{3}}\Sigma_{23},
\end{aligned} \tag{A.4}$$

and use the notation

$$N_- = \tfrac{1}{2\sqrt{3}}N_{22}, \qquad A = A_1. \tag{A.5}$$

The evolution equations (2.8) and the constraint (2.2) for the exceptional Bianchi cosmologies now follow from equations (A.10)–(A.18) in Hewitt *et al* (2001).

Appendix B. The Weyl curvature tensor

In this appendix we give an expression for the Weyl curvature parameter $\mathcal{W}$ in terms of the Hubble-normalized variables Σ_+, Σ_-, Σ_2, $\Sigma_\times$, N_- and A. The Weyl curvature parameter is defined by

$$\mathcal{W}^2 = \frac{E_{ab}E^{ab} + H_{ab}H^{ab}}{6H^4}, \tag{B.1}$$

where E_{ab} and H_{ab} are the electric and magnetic parts of the Weyl tensor, respectively (see WE, p 19), relative to the fluid congruence. Let $E_{\alpha\beta}$ and $H_{\alpha\beta}$ be the components of E_{ab} and H_{ab} relative to the group invariant frame introduced in appendix A. The Hubble-normalized counterparts of $E_{\alpha\beta}$ and $H_{\alpha\beta}$ are defined according to

$$\mathcal{E}_{\alpha\beta} = \frac{E_{\alpha\beta}}{H^2}, \qquad \mathcal{H}_{\alpha\beta} = \frac{H_{\alpha\beta}}{H^2}. \tag{B.2}$$

Since $\mathcal{E}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta}$ are symmetric and trace-free, they each have five independent components. We label these in analogy with the labelling of the shear variables as given in appendix A, i.e.

$$\begin{aligned}
\mathcal{E}_+ &= \tfrac{1}{2}(\mathcal{E}_{22} + \mathcal{E}_{33}), & \mathcal{E}_- &= \tfrac{1}{2\sqrt{3}}(\mathcal{E}_{22} - \mathcal{E}_{33}), \\
\mathcal{E}_\times &= \tfrac{1}{\sqrt{3}}\mathcal{E}_{23}, & \mathcal{E}_2 &= \tfrac{1}{\sqrt{3}}\mathcal{E}_{13}, & \mathcal{E}_3 &= \tfrac{1}{\sqrt{3}}\mathcal{E}_{12},
\end{aligned} \tag{B.3}$$

with similar equations for the frame components of $\mathcal{H}_{\alpha\beta}$. It then follows from (B.1)–(B.3) that

$$\mathcal{W}^2 = \mathcal{E}_+^2 + \mathcal{E}_-^2 + \mathcal{E}_\times^2 + \mathcal{E}_2^2 + \mathcal{E}_3^2 + \mathcal{H}_+^2 + \mathcal{H}_-^2 + \mathcal{H}_\times^2 + \mathcal{H}_2^2 + \mathcal{H}_3^2. \tag{B.4}$$

With the frame choice made in appendix A, equations (1.101) and (1.102) in WE for $E_{\alpha\beta}$ and $H_{\alpha\beta}$, in conjunction with (A.1)–(A.5), lead to

$$
\begin{aligned}
\mathcal{E}_+ &= \Sigma_+(1+\Sigma_+) - \Sigma_-^2 - \Sigma_\times^2 + \tfrac{1}{2}\Sigma_2^2 + 2(N_-^2 + 3A^2), \\
\mathcal{E}_- &= \Sigma_-(1 - 2\Sigma_+) + \tfrac{\sqrt{3}}{2}\Sigma_2^2 + 2\sqrt{3}(N_-^2 - A^2), \\
\mathcal{E}_\times &= \Sigma_\times(1 - 2\Sigma_+) + 8N_- A, \\
\mathcal{E}_2 &= \Sigma_2(1 + \Sigma_+ + \sqrt{3}\,\Sigma_-), \qquad \mathcal{E}_3 = -\sqrt{3}\,\Sigma_2\Sigma_\times,
\end{aligned}
\tag{B.5}
$$

and

$$
\begin{aligned}
\mathcal{H}_+ &= -3\Sigma_- N_- - 3\sqrt{3}\,\Sigma_\times A, \\
\mathcal{H}_- &= -3\Sigma_+ N_- - 2\sqrt{3}\,\Sigma_- N_- + \Sigma_\times A, \\
\mathcal{H}_\times &= -(\Sigma_- + 3\sqrt{3}\,\Sigma_+)A - 2\sqrt{3}\,\Sigma_\times N_-, \\
\mathcal{H}_2 &= \sqrt{3}\,\Sigma_2 N_-, \qquad \mathcal{H}_3 = -4\Sigma_2 A.
\end{aligned}
\tag{B.6}
$$

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